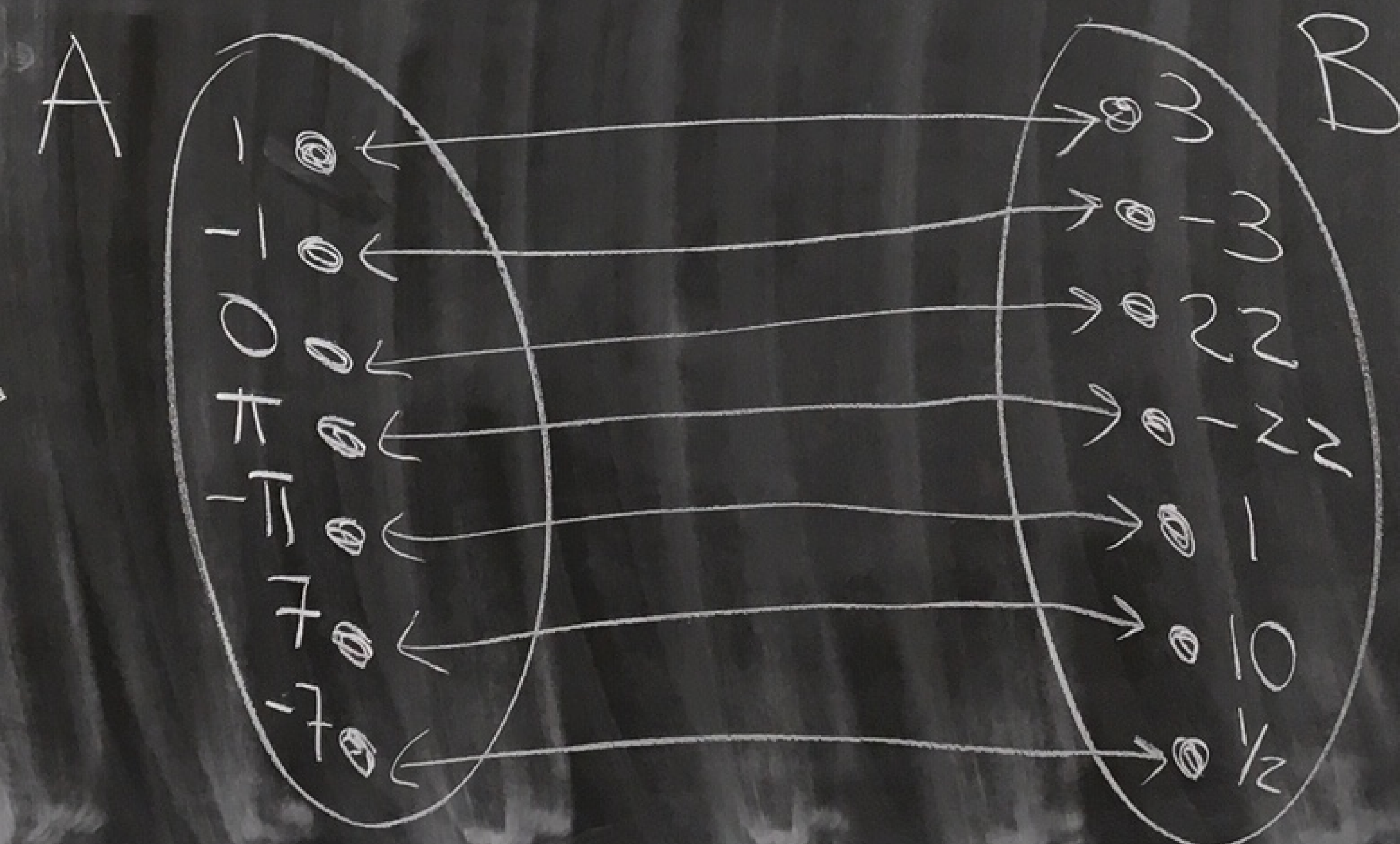
Monday
11/18

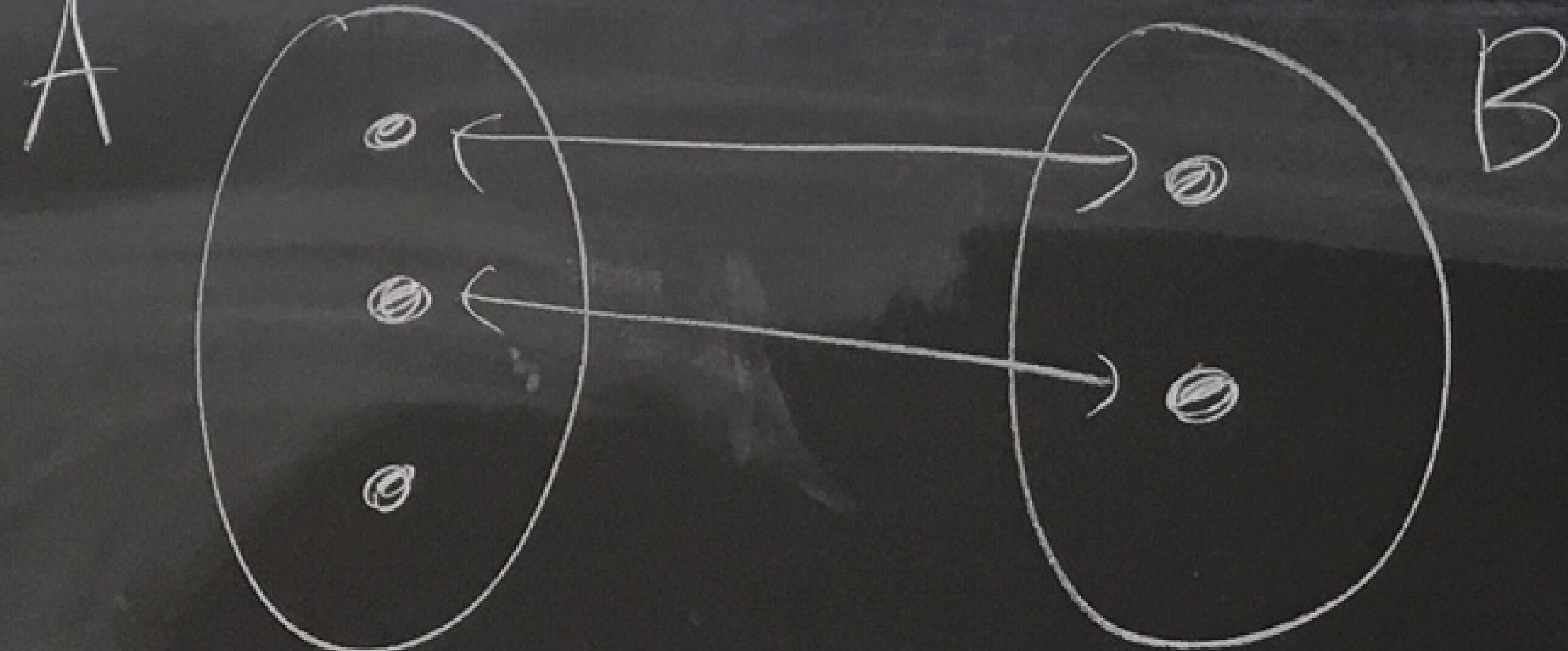
Cardinality

How can we tell if two finite sets have the same number of elements?

Here A and B have the same # of elements



You see if you can pair up the elements in a 1-1 and onto way.



there is no way to pair up the elements of A and B in a 1-1 and onto way.

So, A and B have different sizes.

Def: Let A and B be sets. We say that A and B have the same <u>cardinality</u> if there exists a bijection between them. And if such a bijection exists we write $|A|=|B|$.

If no such bijection exists we say that A and B have <u>unequal cardinality</u> and we write $|A| \neq |B|$.

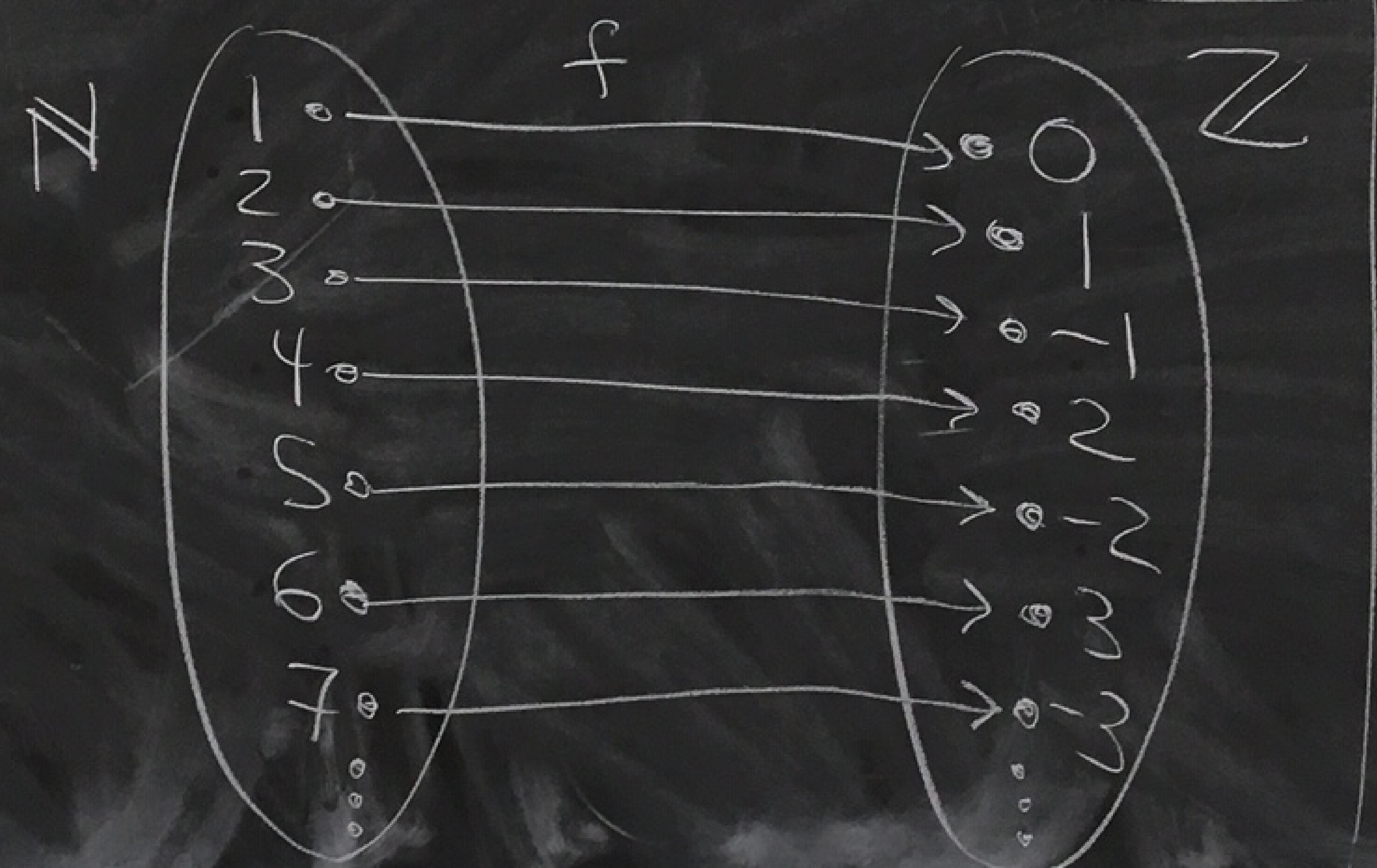
Ex: $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Q: Is $|\mathbb{N}| = |\mathbb{Z}|$ or $|\mathbb{N}| \neq |\mathbb{Z}|$?

$\boxed{\text{Formula for } f}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$



Define $f : \mathbb{N} \to \mathbb{Z}$ as in the picture (and keep going with the same pattern).

Then $f : \mathbb{N} \to \mathbb{Z}$ is a bijec

So, $|\mathbb{N}| = |\mathbb{Z}|$

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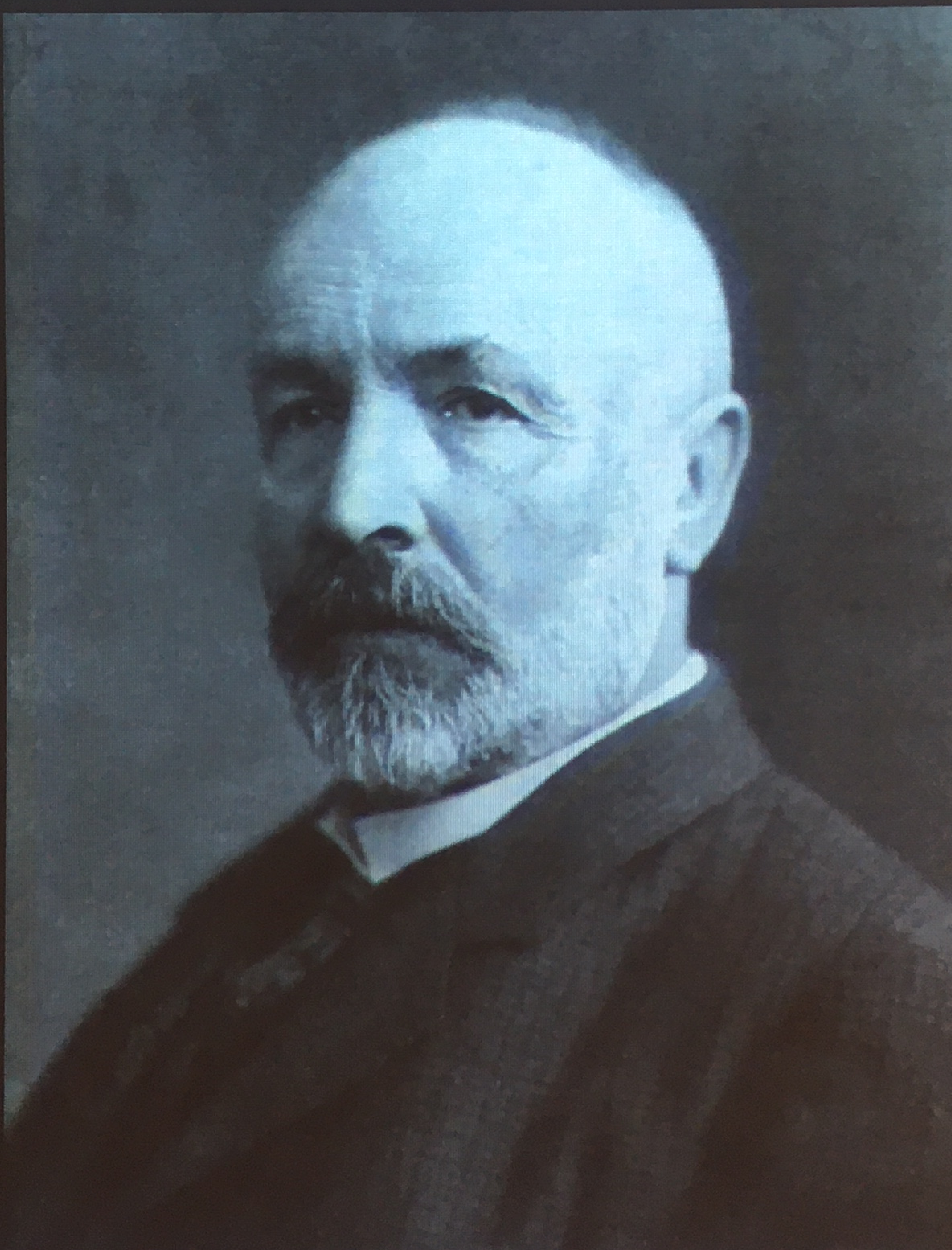

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Ex: $|\mathbb{N}| \neq |\mathbb{R}|$.

(Cantor's Diagonalization Argument)

We will show that there is no bijection $f: \mathbb{N} \to \mathbb{R}$.

Suppose $f: \mathbb{N} \to \mathbb{R}$.
We prove that f can't be onto.

Let's make a table.
Suppose f has the following outputs.

n	$f(n)$
1	$x_1 . b_{11} b_{12} b_{13} b_{14} b_{15} \dots$
2	$x_2 . b_{21} b_{22} b_{23} b_{24} b_{25} \dots$
3	$x_3 . b_{31} b_{32} b_{33} b_{34} b_{35} \dots$
4	$x_4 . b_{41} b_{42} b_{43} b_{44} b_{45} \dots$
5	$x_5 . b_{51} b_{52} b_{53} b_{54} b_{55} \dots$
$\vdots$	$\vdots \quad \ddots$

here $x_i \in \mathbb{Z}$

$b_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$

If the decimal expansion terminates, just add 0's forever.

Define $b = 0.b_1 b_2 b_3 b_4 b_5 \dots$

where $b_i = \begin{cases} 0 & \text{if } b_{ii} \neq 0 \\ 1 & \text{if } b_{ii} = 0 \end{cases}$

By construction $b \neq f(n)$ for all $n \in \mathbb{N}$.

So, b is not in the range of f.

But $b \in \mathbb{R}$. So, f is not onto. $\blacksquare$

Concrete example

n	$f(n)$
1	$1.32547\ldots$
2	$5.0000000\ldots$
3	$-17.3317325\ldots$
4	$0.5000000\ldots$
5	$3.1415926\ldots$
6	$-100.10057892\ldots$
$\vdots$	$\vdots$

$$b_{11}=3 \qquad b_{33}=1 \qquad b_{55}=9$$

$$b = 0.010100\ldots$$

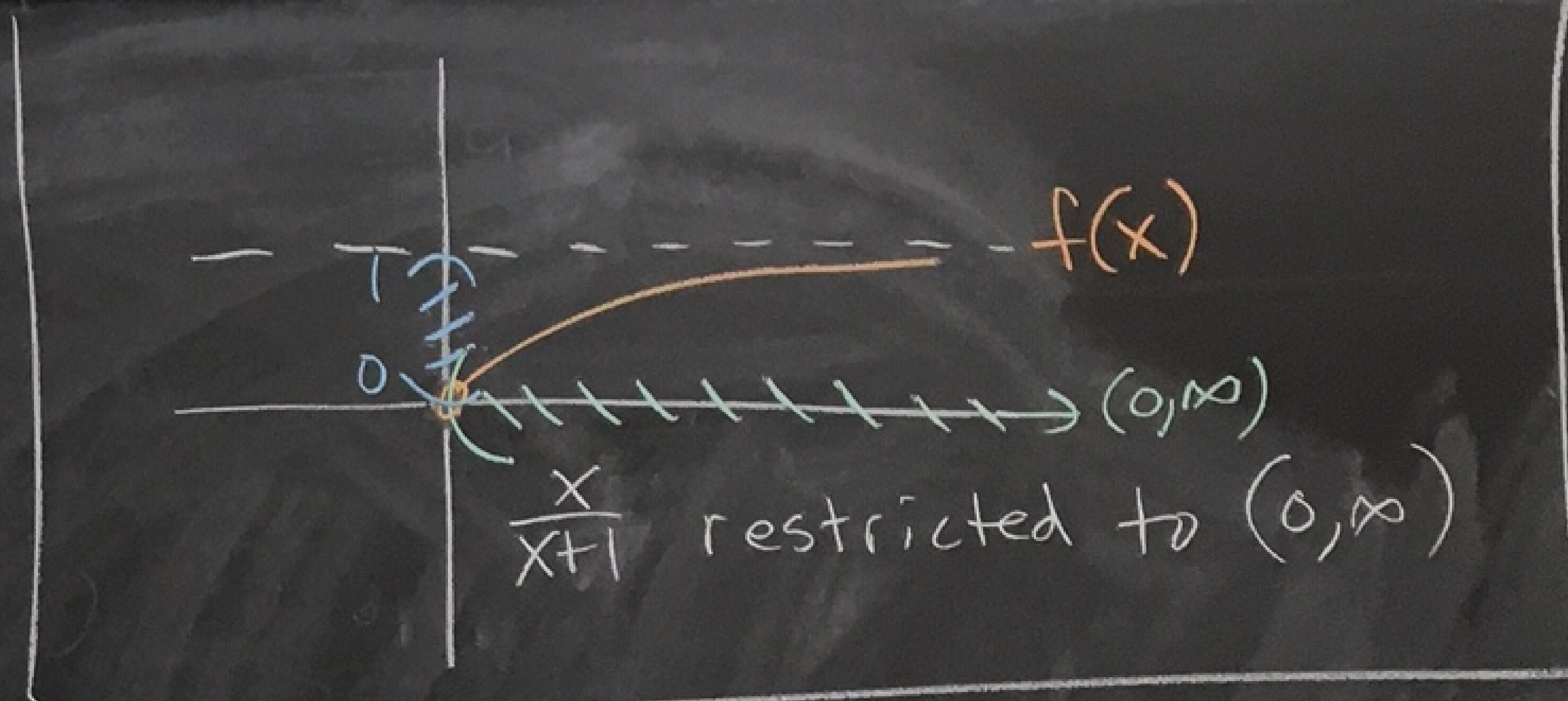
$$b_{22}=0 \qquad b_{44}=0 \qquad b_{66}=8$$

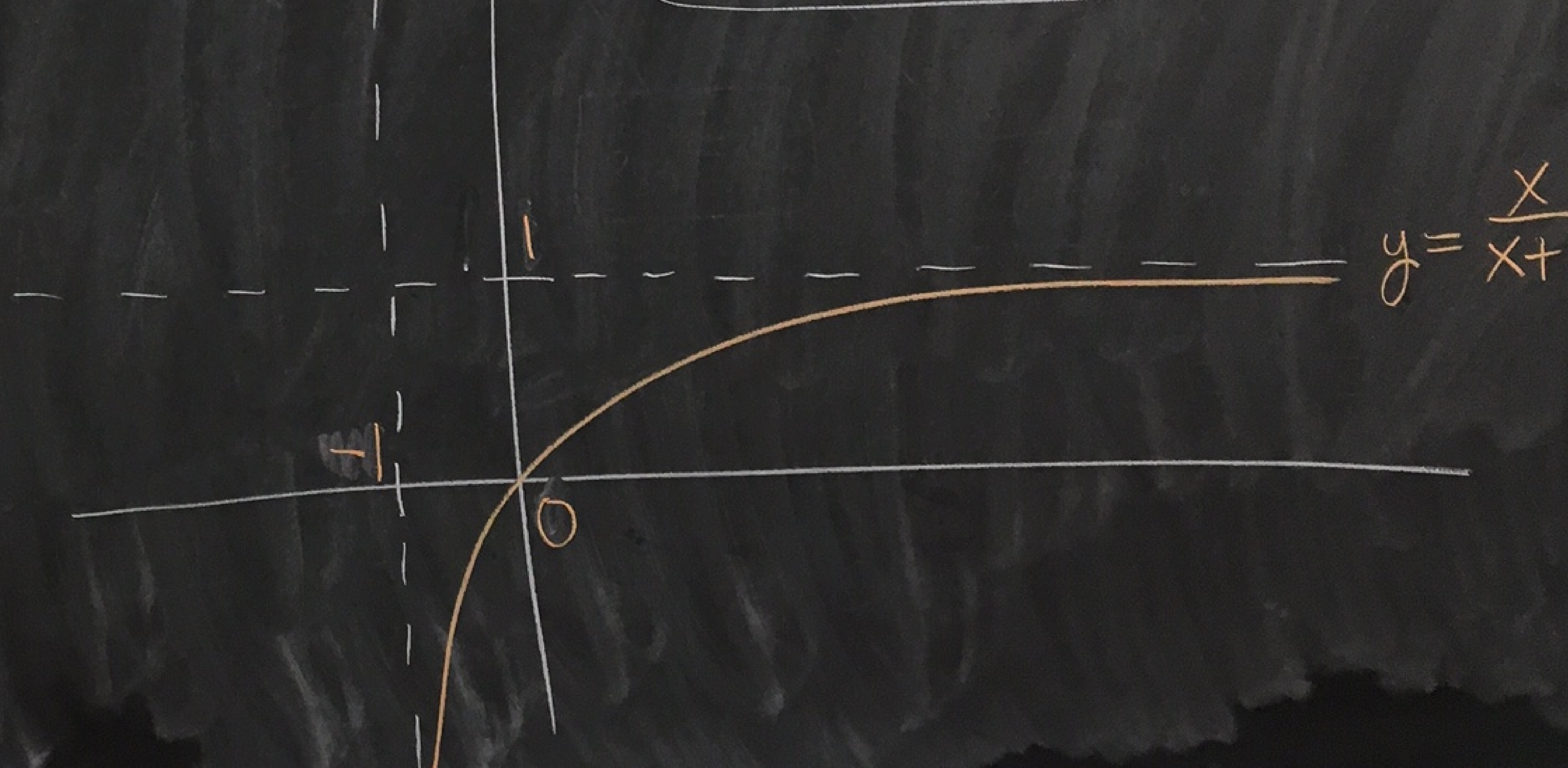
Ex: $|(0,\infty)| = |(0,1)|$

Let $f:(0,\infty) \longrightarrow (0,1)$

be defined by

$$f(x) = \frac{x}{x+1}$$





f is 1–1

Suppose $f(x_1) = f(x_2)$ where $x_1, x_2 \in (0, \infty)$.

Then $\dfrac{x_1}{x_1+1} = \dfrac{x_2}{x_2+1}$.

So, $x_1 x_2 + x_1 = x_1 x_2 + x_2$.

Thus, by adding $-x_1 x_2$ to both sides,

we get $x_1 = x_2$.

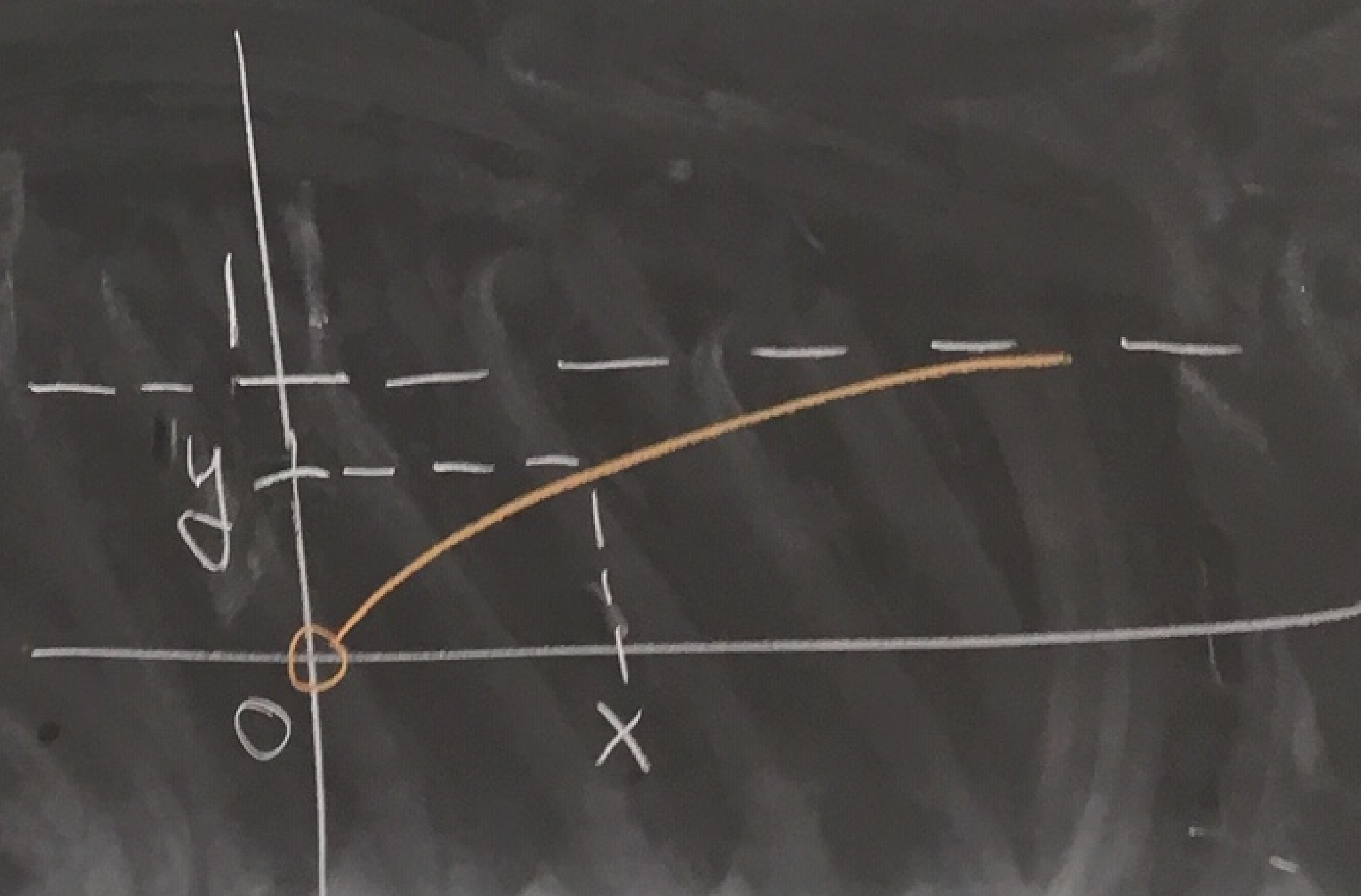
f is onto

Let $y \in (0,1)$.

We want to find

$x \in (0, \infty)$ where

$f(x) = y$.



Let's solve $\frac{x}{x+1} = y$.

We want $x = xy + y$,

ie $x - xy = y$,

ie $x(1-y) = y$.

ie $\boxed{x = \frac{y}{1-y}}$

<u>Check:</u> $$f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{\left(\frac{y}{1-y}\right)+1} = \frac{\left(\frac{y}{1-y}\right)}{\left(\frac{y+(1-y)}{1-y}\right)} = \frac{y}{1} = y.$$

Is $x \in (0,\infty)$?

We know $0 < y < 1$.

So, $-1 < -y < 0$.

Thus, $0 < 1-y < 1$.

Thus, $\boxed{0 < \frac{y}{1-y} = x}$.

<u>Summary:</u> Given $y \in (0,1)$, set $x = \frac{y}{1-y}$. Then $f(x) = y$ and $0 < x < \infty$. So, f is onto. ▨

$$\boxed{\text{So, } |(0,\infty)| = |(0,1)|}$$

Monday
12/2

Recall from last time

We say that two sets A and B have the <u>same cardinality</u> if there exists a bijection between them. If so, we write $|A| = |B|$.
If no such bijection exists we write $|A| \neq |B|$.

Ex:

$|\mathbb{N}| = |\mathbb{Z}|$

$|\mathbb{N}| \neq |\mathbb{R}|$

$|(0,1)| = |(0,\infty)|$

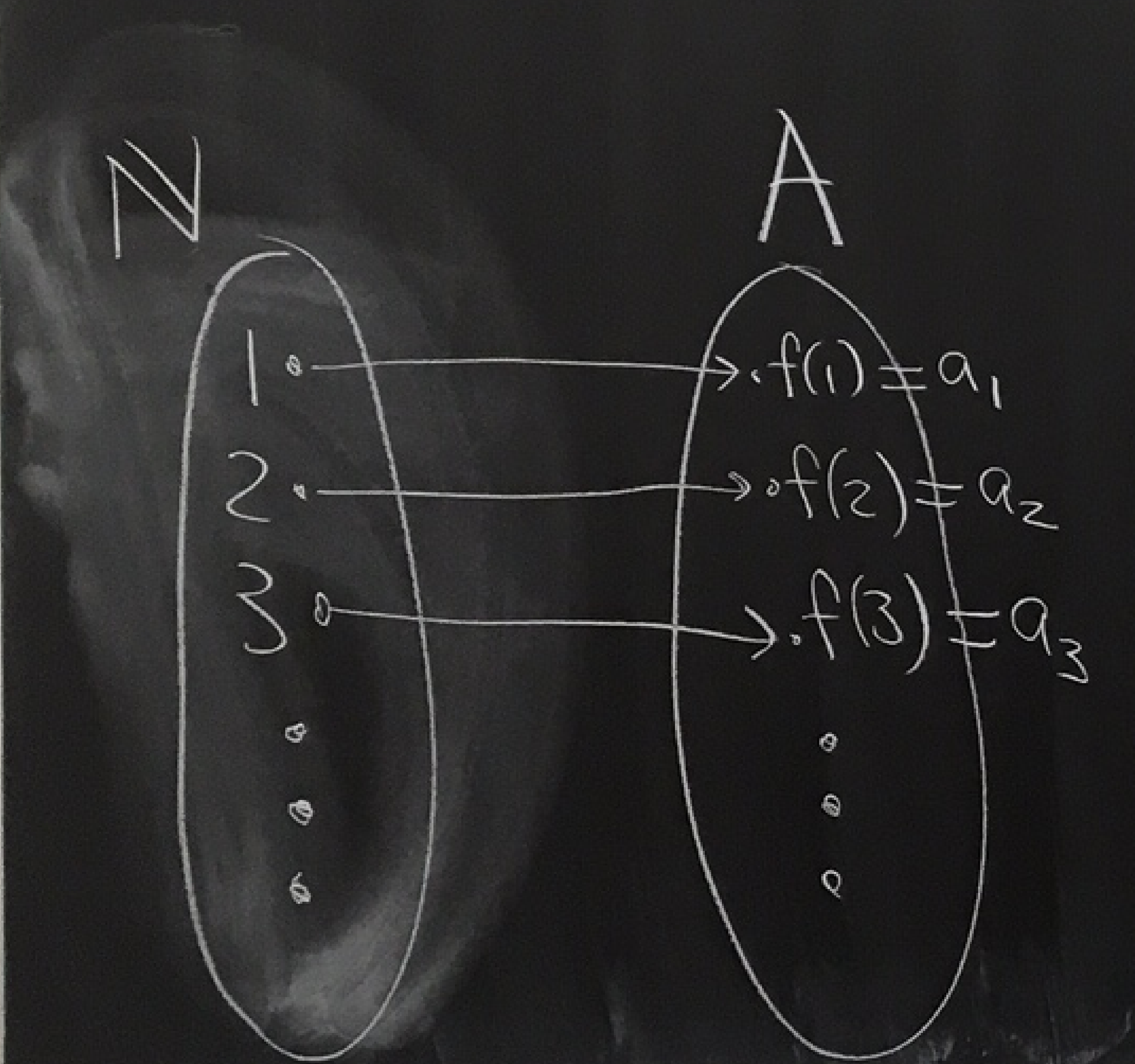
Def: Let A be a set.

- We say that A is <u>countably infinite</u> if $|A| = |\mathbb{N}|$
- We say that A is <u>countable</u> if either A is finite or A is countably infinite.
- We say that A is <u>uncountable</u> if A is not countable, that is A is infinite and $|A| \neq |\mathbb{N}|$.

Ex: $\mathbb{Z}$ is countably infinite. $\mathbb{Z}$ is countable.

$\{1, 2, 4\}$ is countable. $\mathbb{R}$ is uncountable.

Theorem: A set A is countably infinite iff its elements can be arranged in an infinite list $a_1, a_2, a_3, \ldots$ with no repeats.



proof:

($\Rightarrow$) Suppose A is countably infinite. Then there exists $f: \mathbb{N} \to A$ that is a bijection. Define $a_i = f(i)$.

Then consider the list $a_1, a_2, a_3, \ldots$

Since f is onto, our list gives all of A.

Since f is 1-1, the list has no repeats.

($\Leftarrow$) Suppose the elements of A can be arranged in an infinite list $a_1, a_2, a_3, \ldots$ with no repeats.

Define $f: \mathbb{N} \to A$ by $f(i) = a_i$.

Since the list has no repeats, f is 1-1.

Since the list contains all the elements of A, f is onto.

So f is a bijection and $|\mathbb{N}| = |A|$.

So, A is countably infinite. $\blacksquare$

Theorem: $\mathbb{Q}$ is countably infinte.

Start here

0	1	-1	2	-2	3	-3	...
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	...
	$\frac{1}{2}$	$\frac{-1}{2}$	$\cancel{\frac{2}{2}}$	$\cancel{\frac{-2}{2}}$	$\frac{3}{2}$	$\frac{-3}{2}$	...
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\cancel{\frac{3}{3}}$	$\cancel{\frac{-3}{3}}$	...
	$\frac{1}{4}$	$\frac{-1}{4}$	$\cancel{\frac{2}{4}}$	$\cancel{\frac{-2}{4}}$	$\frac{3}{4}$	$\frac{-3}{4}$	...
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{5}$	$\frac{-3}{5}$	...
	$\frac{1}{6}$	$\frac{-1}{6}$	$\cancel{\frac{2}{6}}$	$\cancel{\frac{-2}{6}}$	$\cancel{\frac{2}{6}}$	$\cancel{\frac{-3}{6}}$	...
	⋮	⋮	⋮	⋮	⋮	⋮	...

0	1	−1	2	−2	3	−3	4	−4	5	−5	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{3}{11}$	$\frac{-3}{11}$	$\frac{4}{15}$	$\frac{-4}{15}$	$\frac{5}{9}$	$\frac{-5}{9}$	$\cdots$
	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{2}{17}$	$\frac{-2}{17}$	$\frac{3}{13}$	$\frac{-3}{13}$	$\frac{4}{17}$	$\frac{-4}{17}$	$\frac{5}{11}$	$\frac{-5}{11}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

We put $\mathbb{Q}$ into an infinite list with no repeats by weaving through the infinite table as indicated and removing repeats as we encounter them. Here are the first entries in the list:

$$0, 1, \frac{1}{2}, -\frac{1}{2}, -1, 2, \frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \ldots$$

This will give an arrangement of $\mathbb{Q}$ into an infinite list with no repeats. So, $\mathbb{Q}$ is countably infinite. $\boxed{/\!/}$

sets in each column have same cardinality

Countably infinite	Uncountable			
$\mathbb{N}$	$\mathbb{R}$	$\mathcal{P}(\mathbb{R})$	$\mathcal{P}(\mathcal{P}(\mathbb{R}))$	$\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$ ...
$\mathbb{Z}$	$\mathcal{P}(\mathbb{N})$			
$\mathbb{Q}$				

$\aleph$ is the first letter in the Hebrew alphabet

aleph

this cardinality is called $\aleph_0$ aleph naught

We will show $\mathcal{P}(A)$ is "bigger" than A for all A.

is there any set "in between" in cardinality between $\mathbb{N}$ and $\mathbb{R}$?

unknown.

Continuum hypothesis

You can keep getting "bigger" and "bigger" infinite sets by continually taking the power set of each set.

Some more theorems (see Hammack)

<u>Thm:</u> If A and B are countably infinite, then $A \times B$ is countably infinite. [Similar proof as $\mathbb{Q}$ is countably infinite.]

<u>Thm:</u> If A and B are countably infinite, then $A \cup B$ is countably infinite.

<u>Thm:</u> An infinite subset of a countably infinite set is countably infinite.

<u>Thm:</u> If $U \subseteq A$ and U is uncountable, then A is uncountable.

<u>Thm:</u>

$$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$$

12/4
Weds

Def: Let A and B be sets.

① $|A| = |B|$ means there exists a bijection $f: A \to B$.

② $|A| \leq |B|$ means there exists a one-to-one function $f: A \to B$

③ $|A| < |B|$ means there exists a one-to-one function $f: A \to B$, but there is no bijection $g: A \to B$

Ex: $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

$|\mathbb{N}| < |\mathbb{R}|$

The next theorem will show that

$$|\mathbb{N}| < |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))| < \cdots$$

So there is no "biggest" set.

Theorem: Let A be a set. Then $|A| < |\mathcal{P}(A)|$.

pf:

If A is finite then $|A| < 2^{|A|} = |\mathcal{P}(A)|$.

Now assume A is infinite.

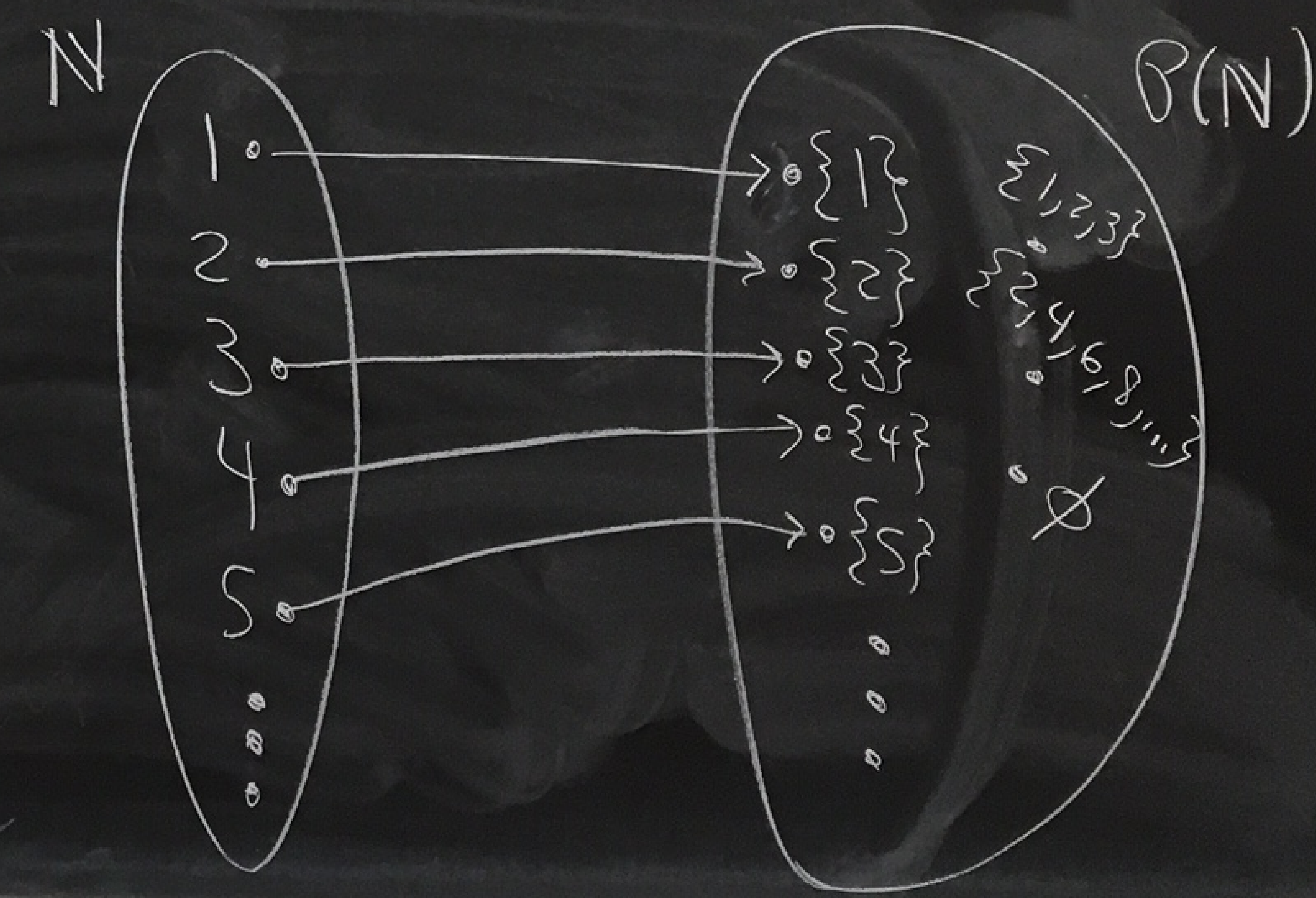
Step 1: Create $f: A \to \mathcal{P}(A)$ that is 1-1

Define $f: A \to \mathcal{P}(A)$ by $f(a) = \{a\}$.

Let's show f is 1-1.

Suppose $f(x) = f(y)$ where $x, y \in A$.

Side example of $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ that is 1-1.



Then $\{x\}=\{y\}$. $\leftarrow$ $\boxed{\begin{array}{l} x\in\{y\} \\ \text{So } x=y. \end{array}}$

So, $x=y$.

So, $|A| \le |\mathcal{P}(A)|$.

Step 2: Show there is no bijection $g: A \to \mathcal{P}(A)$.

Suppose $g: A \to \mathcal{P}(A)$.

We will show g cannot be onto.

We do this by constructing an element $B \in \mathcal{P}(A)$ that isn't in the range of g.

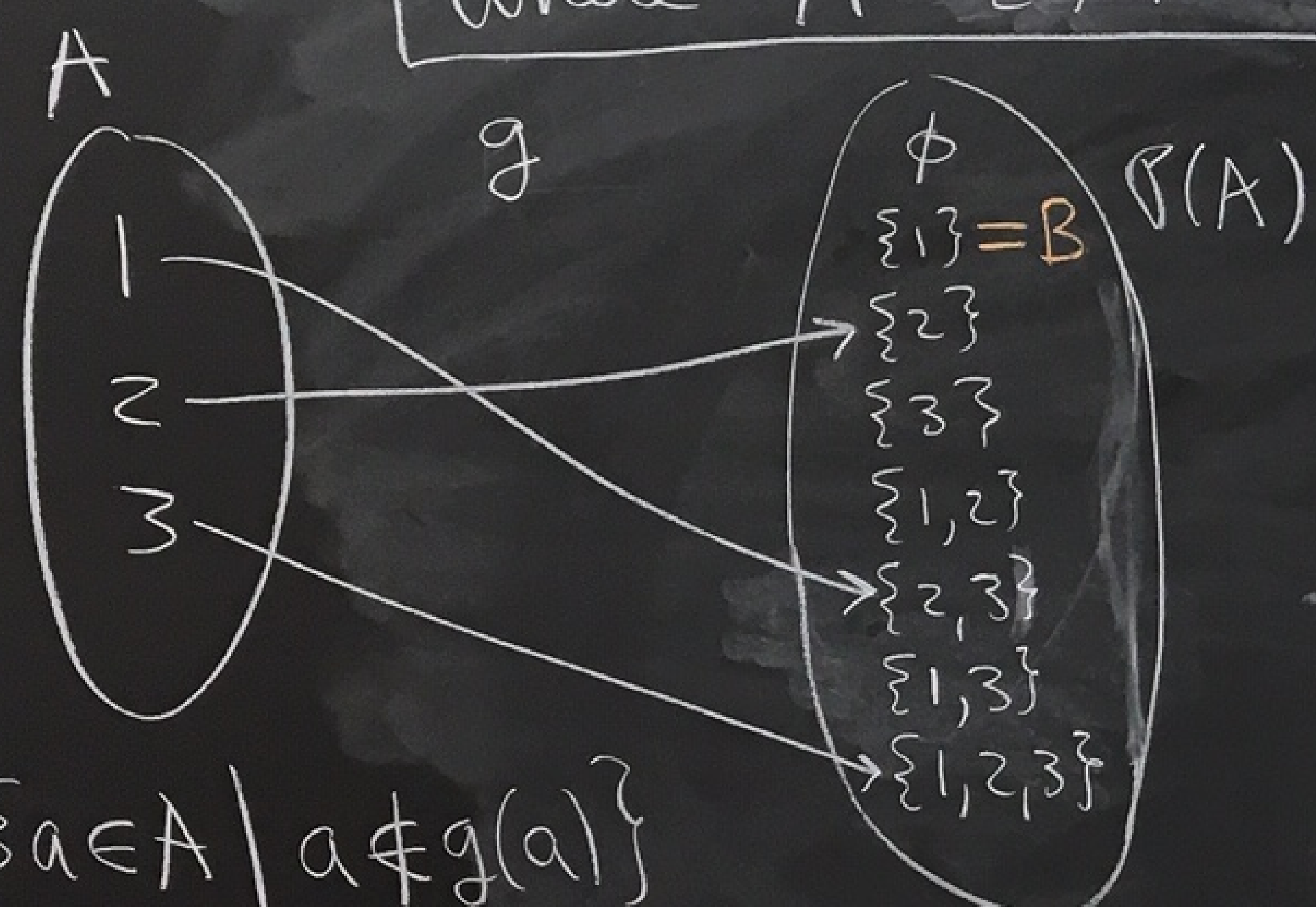
Let

$$B = \{a \in A \mid a \notin g(a)\}.$$

Note $B \subseteq A$, so $B \in \mathcal{P}(A)$.

Let's show $B \notin \text{range}(g)$.

Example of B where $A=\{1,2,3\}$



$B=\{a\in A \mid a\notin g(a)\}$

$1\notin g(1)$? Yes. $1\notin\{2,3\}$

$2\notin g(2)$? No. $2\in\{2\}$

$3\notin g(3)$? No. $3\in\{1,2,3\}$

$B=\{1\}$

$B\notin \text{range}(g)$

(proof continued...)

Let $a\in A$.

We will show $g(a)\neq B$.

One we've shown this then we know there is no element of A that maps to B.

case 1: Suppose $a\in g(a)$.

Thus, by the def. of B, $\boxed{a\notin B}$.

Why can't we have $g(a)=B$?

Suppose $g(a)=B$.

By assumption, $a\in g(a)$. So if $g(a)=B$, then $\boxed{a\in B}$.

B.

$a \in B.$

Then we would have both $a \notin B$ and $a \in B$, which can't happen.

So in this case, $g(a) \neq B$.

Case 2: Suppose $a \notin g(a)$

Since $a \notin g(a)$ we have $\boxed{a \in B.}$

Why can't $g(a) = B$?

If $g(a) = B$, then since $a \notin g(a)$ we would get $\boxed{a \notin B.}$

We can't have $a \in B$ and $a \notin B$.

So in this case, $g(a) \neq B$.

In both cases $g(a) \neq B$.

Since a was an arbitrary element of A, we have shown $B \notin \text{range}(g)$.

So, g isn't onto. $\blacksquare$

Cantor-Bernstein-Schröder Theorem

Let A and B be sets.

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

That is, if there exist one-to-one functions $f: A \to B$ and $g: B \to A$, then there exists a bijection $h: A \to B$.

Thm: $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

$$\begin{array}{ccccccc} |\mathbb{Q}| & & & & & & \\ \shortparallel & & & & & & \\ |\mathbb{N}| & < & |\mathbb{R}| & < & |\mathcal{P}(\mathbb{R})| & < & \cdots \\ \shortparallel & & \shortparallel & & & & \\ |\mathbb{Z}| & & |\mathcal{P}(\mathbb{N})| & & & & \end{array}$$